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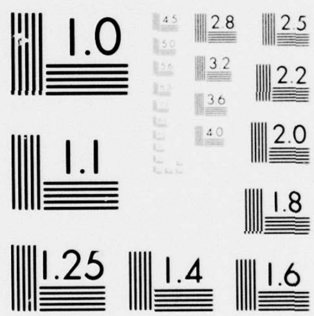

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The Existence of Almost Surely Comparable Versions of Stochastically Comparable Random Vectors and Functions

by

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THE EXISTENCE OF ALMOST SURELY COMPARABLE VERSIONS OF STOCHASTICALLY COMPARABLE RANDOM VECTORS AND FUNCTIONS[1]

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Abstract

R(n), R(inf.), X and Y are random elements of a function space, for example, R^n, R^∞, $C(T)$, or $D(T)$, such that X is stochastically less than Y. Then there exist random elements X' and Y' defined on a common probability space distributed the same as X and Y respectively, such that X' is almost surely less than Y'.

1. Introduction and Summary.

Let S be a space with a partial order, $<<$. A lower set $L \subset S$ satisfies: if $s \in L$ and $t << s$ then $t \in L$. Let X and Y be random elements of S; X is stochastically less than Y (denoted $X \overset{st}{\leq} Y$) if $P(X \in L) \geq P(Y \in L)$ for all measurable lower sets L of S. If X and Y are defined on the same probability space (Ω, F, P); i.e. X and Y are measurable functions mapping (Ω, F, P) into S, then X is almost surely less than Y (denoted $X \overset{as}{\leq} Y$) if $P(\omega \in \Omega : X(\omega) << Y(\omega)) = 1$.

Comparison techniques are frequently used in situations which are too complicated for precise analysis and which are not suited to asymptotic analysis. Kalmykov(1962), Veinott(1965), Daley(1968), Pledger and Proschan (1973) and Keilson and Kester(1974) look at various aspects of stochastic comparison. O'Brien(1972,1975a) and Sonderman and Whitt(1976) consider almost sure comparison. There is frequent application of these concepts in the literature: Jacobs and Schach(1972), Stoyan(1973), O'Brien(1975b) and others compare queues. Pledger and Proschan(1973), Ross(1974), Keilson(1974) and Barlow and Proschan(1976) make comparisons of reliability systems. Proschan and Sethuraman(1975) compare order statistics from different populations. Many more cases of comparisons can be found in the literature.

In the case of real-valued random variables the following relationship between stochastic inequality and almost sure inequality is well known: If $X \overset{st}{\leq} Y$ then there exists a probability space (Ω, F, Q) and random variables X' and Y' on that space such that $X' \overset{as}{\leq} Y'$ and $X' \overset{D}{=} X$ and $Y' \overset{D}{=} Y$, where "$\overset{D}{=}$" signifies that the random variables have the same distribution. (If $F(x) = P(X \leq x)$ and $G(y) =$

$P(Y \leq y)$ then let $\Omega = [0,1]$, F = Borel sets, $X'(\omega) = F^{-1}(\omega)$, and $Y'(\omega) = G^{-1}(\omega)$.) The purpose of this paper is to extend this result to higher dimensions. There are two motivations for doing this: First, stochastic and almost sure comparison are important and useful concepts; it is of general interest to discover relationships between them. Secondly, a useful relationship may be a valuable technique in establishing or exploiting inequalities. An analogous situation occurs in the theory of weak convergence where existence of almost surely convergent versions of weakly convergent sequences of measures is a powerful tool; see Skorokhod (1956), Dudley (1968) and Pyke (1969).

This paper is arranged as follows: In section 2 the existence is proven of almost surely comparable versions of stochastically comparable random vectors of N^k, the k-fold product of copies of the natural numbers $\{1, 2, 3,...\}$. In section 3 the result is proven for stochastically comparable random elements of $D(T)$, the space of real-valued functions on a bounded or unbounded interval T which are left-continuous and have limits from the right. As special cases, the result will hold for random elements of $C(T)$, R^∞ and R^n. In section 4 an equivalent definition of stochastic inequality is discussed.

2. Comparable Random Vectors in N^k.

Let N be the natural numbers, $\{1, 2, 3,...\}$, and N^k the k-fold Cartesian product. Define a partial order on N^k as: for $x = (x_1, x_2, ..., x_k)$ and $y = (y_1, y_2, ..., y_k)$, $x << y$ if $x_i \leq y_i$ for $i = 1, 2, ..., k$. This is the usual partial ordering for real vectors.

<u>Theorem 1</u>. Let X and Y be random vectors in N^k such that $X \overset{st}{\leq} Y$ Then there exists a probability space (Ω, F, Q) and measurable functions $X', Y'\colon\ (\Omega, F, Q) \to N^k$ such that $X' \overset{D}{=} X$ $Y' \overset{D}{=} Y$ and $X' \overset{a.s.}{\leq} Y'$.

Before proving Theorem 1, it will be convenient to state and prove a lemma.

<u>Lemma 1</u>. ξ is a signed measure of bounded variation on a family of subsets L which is closed under countable unions and intersections. If $\inf_{L \in L} \xi(L) = a$, then there exists $L_o \in L$ such that $\xi(L_o) = a$.

<u>Proof of Lemma 1</u>. There must exist a sequence $\{L_n, n = 1, 2, \dots\}$ such that $\xi(L_n) \downarrow a$. From this sequence we shall construct the desired set L_o. We shall construct a triangular array of sets in L:

$$\begin{array}{ccccccccc}
L_{1,1} & & & & & & & & \\
\cap & & & & & & & & \\
L_{2,1} & \supset & L_{2,2} & & & & & & \\
\cap & & \cap & & & & & & \\
L_{3,1} & \supset & L_{3,2} & \supset & L_{3,3} & & & & \\
\cap & & \cap & & \cap & & & & \\
 & \dots & & & & & & & \\
L_{\infty,1} & \supset & L_{\infty,2} & \supset & L_{\infty,3} & \supset & L_{\infty,4} & \supset & \dots
\end{array} \tag{2.1}$$

such that

$$L_{\infty,i} = \bigcup_{n=1}^{\infty} L_{n,i}\,, \qquad i = 1, 2, \dots \tag{2.2}$$

$$\xi(L_{n,n}) \leq \xi(L_n)\,, \qquad n = 1, 2, \dots \tag{2.3}$$

$$\xi(L_{n+1,i}) \le \xi(L_{n,i}), \qquad i \ge n, \quad n = 1,2,\ldots \tag{2.4}$$

If such a triangular array can be constructed then from (2.2), (2.3) and (2.4) it follows that $\xi(L_{\infty,i}) \le \xi(L_i)$ and thus $\overline{\lim_{i\to\infty}} \xi(L_{\infty,i}) \le \lim_{i\to\infty} \xi(L_i) =$ a. Since $L_{\infty,i} \in \mathcal{L}$, $\xi(L_{\infty,i}) \ge a$ and consequently $\lim_{i\to\infty} \xi(L_{\infty,i}) = a$. Since ξ is of bounded variation $\xi(\bigcap_{i=1}^{\infty} L_{\infty,i}) = a$ and thus $L_o = \bigcap_{i=1}^{\infty} L_{\infty,i}$ is the sought-after set in $\mathcal{L}$.

We now verify the existence of the triangular array (2.1) which satisfies (2.2), (2.3) and (2.4). We use a two-stage induction (on rows and columns): Define $L_{1,1} = L_1$; this establishes the first row of (2.1). Assume that (2.1) is constructed through the nth row:

$$L_{n,1} \subset L_{n,2} \subset \cdots \subset L_{n,n} \tag{2.5}$$

We shall now construct the $n+1$st row: Define the sequence $\{L_{n+1}^i, i = 1, 2, \ldots, n\}$ by induction as follows:

$$L_{n+1}^1 = \begin{cases} L_{n+1} & \text{if} \quad \xi(L_{n+1} - L_{n,1}) < 0 \\ L_{n+1} \cap L_{n,1} & \text{if} \quad \xi(L_{n+1} - L_{n,1}) \ge 0 \end{cases}$$

$$L_{n+1}^i = \begin{cases} L_{n+1}^{i-1} & \text{if} \quad \xi(L_{n+1}^{i-1} - L_{n,i}) < 0 \\ L_{n+1}^{i-1} \cap L_{n,i} & \text{if} \quad \xi(L_{n+1}^{i-1} - L_{n,i}) \ge 0 \end{cases}$$

for $i = 2, 3, \ldots, n$, and

$$L_{n+1}^{n+1} = \bigcap_{i=1}^{n} L_{n+1}^{i}.$$

Thus

$$L_{n+1}^1 \supset L_{n+1}^2 \supset \cdots \supset L_{n+1}^n = L_{n+1}^{n+1} \tag{2.6}$$

and

$$\xi(L_{n+1}) \geq \xi(L^1_{n+1}) \geq \xi(L^2_{n+1}) \geq \ldots \geq \xi(L^{n+1}_{n+1}) \qquad (2.7)$$

Define $L_{n+1,i} = L_{n,i} \cup L^i_{n+1}$, $i = 1, 2, \ldots, n$. Condition (2.4) then follows from the definition of L^i_{n+1}. From (2.5) and (2.6) it follows that $L_{n+1,i} \supset L_{n+1,i+1}$. Define $L_{n+1,n+i} = L^{n+1}_{n+1}$. Condition (2.3) follows from (2.7). Condition (2.2) follows by defining $L_{\infty,i}$ accordingly. This verifies the existence of (2.1) and completes the proof of Lemma 1.

<u>Proof of Theorem 1</u>. Let μ be the measure induced by X on N^k; let ν be the measure induced by Y. Then for all lower sets L, $\mu(L) \geq \nu(L)$. We shall say that μ is stochastically less than ν, $\mu \overset{st}{\leq} \nu$. The theorem is proven by constructing a sequence of measures $\{\mu_n, n = 1, 2, \ldots\}$ on N^k such that $\mu \overset{st}{\leq} \mu_1 \overset{st}{\leq} \mu_2 \leq \ldots \overset{st}{\leq} \nu$ and $\mu_n \Rightarrow \nu$, as $n \to \infty$.

Enumerate the points of N^k in such a way that if $x << y$ then x comes before y is the enumeration. The "first" point of N^k will be $x_1 = (1, 1, 1, \ldots, 1)$. There are many different ways to continue with the enumeration. A suitable generalization of the well-known "diagonal enumeration" of N^2 ([5], p. 60) is one way. In particular, if $\pi_i: N^k \to N$ is the projection onto the i coordinate, let $D_m = \{x \in N^k: \sum_{i=1}^{k} \pi_i(x) = m\}$; then if $y \in D_n$ and $x \lneq y$ it follows that $x \in D_m$, $m<n$. Thus the points of N^k can be enumerated by enumerating the points within each D_n in an arbitrary manner and then combining into one enumeration such that the points of D_m precede the points of D_{m+1}, $m = k, k+1, \ldots$.

We shall construct μ_1 to satisfy:

$$\mu \overset{st}{\leq} \mu_1 \overset{st}{\leq} \nu \tag{2.8}$$

and

$$\mu_1(\{x_1\}) = \nu(\{x_1\}) . \tag{2.9}$$

Define the unit vectors $\{\varepsilon_{i,}\ i = 1, \ldots, k\}$: $\varepsilon_1 = (1, 0, 0, \ldots, 0)$, $\varepsilon_2 = (0, 1, 0, \ldots, 0)$, etc.

Let $\mathcal{L}$ be the collection of all lower sets of N^k and

$$\mathcal{L}_{1,1} = \{ L \in \mathcal{L} : x_1 \in L , \ x_1 + \varepsilon_1 \notin L \}$$

Consider

$$\inf_{\mathcal{L}_{1,1}} (\mu(L) - \nu(L)) = a_{1,1} . \tag{2.10}$$

By Lemma 1 there exists a lower set $L_{1,1}$ in $\mathcal{L}_{1,1}$ such that $\mu(L_{1,1}) = \nu(L_{1,1}) + a_{1,1}$. Let δ_x denote the point probability measure which has unit mass at x. Define $\mu_{1,1} = \mu + a_{1,1} (\delta_{x_1+\varepsilon_1} - \delta_{x_1})$, thus $\mu_{1,1}$ is constructed from μ by shifting mass $a_{1,1}$ from x_1 to $x_1 + \varepsilon_1$. Clearly $\mu(L) \geq \mu_{1,1}(L)$ for any $L \in \mathcal{L}_{1,1}$ and $\mu(L) = \mu_{1,1}(L)$ for any $L \in \mathcal{L} - \mathcal{L}_{1,1}$, thus $\mu \overset{st}{\leq} \mu_{1,1}$. By (2.10) it follows that $\mu_{1,1}(L) - \nu(L) \geq 0$ for $L \in \mathcal{L}_{1,1}$ and from above $\mu_{1,1}(L) = \mu(L) \geq \nu(L)$ for any $L \in \mathcal{L} - \mathcal{L}_{1,1}$. Thus

$$\mu \overset{st}{\leq} \mu_{1,1} \overset{st}{\leq} \nu \ , \quad \mu_{1,1}(L_{1,1}) = \nu(L_{1,1}) .$$

Now define

$$\mathcal{L}_{1,2} = \{ L \in \mathcal{L} : L \subseteq L_{1,1} , x_1 \in L , x_1 + \varepsilon_2 \notin L \}$$

Consider

$$\inf_{\mathcal{L}_{1,2}} (\mu_{1,1}(L) - \nu(L)) = a_{1,2} .$$

As before there exists a lower set $L_{1,2}$ in $\mathcal{L}_{1,2}$ such that $\mu_{1,1}(L_{1,2}) = \nu(L_{1,2}) + a_{1,2}$. Define $\mu_{1,2} = \nu_{1,1} + a_{1,2} (\delta_{x_1+\varepsilon_2} - \delta_{x_1})$. Again, as above, $\mu_{1,1} \overset{st}{\leq} \mu_{1,2} \overset{st}{\leq} \nu$ and $\mu_{1,2}(L_{1,2}) = \nu(L_{1,2})$. Continuing inductively, let

$$\mathcal{L}_{1,i} = \{ L \in \mathcal{L} : L \subseteq L_{1,i-1} , x_1 \in L , x_1 + \varepsilon_i \notin L \}$$

and consider

$$\inf_{\mathcal{L}_{1,i}} (\mu_{1,i-1}(L) - \nu(L)) = a_{1,i} .$$

Define $\mu_{1,i} = \mu_{1,i-1} + a_{1,i} (\delta_{x_1+\varepsilon_i} - \delta_{x_1})$. Then $\mu_{1,i-1} \overset{st}{\leq} \mu_{1,i} \overset{st}{\leq} \nu$ and there exists $L_{1,i} \in \mathcal{L}_{1,i}$ such that $\mu_{1,i}(L_{1,i}) = \nu(L_{1,i})$. Continue until $i = k$. By definition $\mathcal{L}_{1,k} \subset \{L \in \mathcal{L}: x_1 \in L, x_, + \varepsilon_i \notin L, i = 1, 2, \ldots, k\}$ thus $\mathcal{L}_{1,k} = \{\{x_i\}\}$. Define $\mu_1 = \mu_{1,k}$ then (2.8) and (2.9) are satisfied.

Now we shall describe the construction of μ_1 from μ in terms of a function $g_1: N^k \times [0,1) \to N^k [0,1)$. Let λ be Lebesque measure on $[0,1)$. Define

$$g_1(x_1,z) = \begin{cases} \left(x_1 + \varepsilon_i \,,\ \dfrac{z - 1 + \sum\limits_{j=1}^{i} a_{1,j}}{a_{1,i}}\right), & 1-\sum\limits_1^i a_{1,j} \le z < 1 - \sum\limits_1^{i-1} a_{1,j} \\[2ex] \left(x_1 \,,\ \dfrac{z}{1 - \sum\limits_{j=0}^{k} a_{1,j}}\right), & 0 \le z < 1 - \sum\limits_1^k a_{1,j} \end{cases}$$

$$g_1(x,z) = (x,z) \ , \quad x \neq x_1 \ .$$

Then $(\mu \times \lambda)g_1^{-1} = (\mu_1 \times \lambda)$ or equivalently $\mu_1 = (\mu \times \lambda)(\pi_{N^k} \circ g_1)^{-1}$. Close examination of the function g_1 reveals that it is an equivalent way of describing analytically the "shifting of mass" of the preceeding paragraphs.

We shall now proceed inductively to construct a sequence of measures $\{\mu_i, \ i = 1, 2, \ldots\}$ which satisfy:

$$\mu \overset{st}{\le} \mu_1 \overset{st}{\le} \mu_2 \overset{st}{\le} \ldots \overset{st}{\le} \nu \tag{2.11}$$

$$\mu_n(\{x_m\}) = \nu(\{x_m\}) \quad , \quad m \le n \tag{2.12}$$

The construction of μ_n from μ_{n-1} is the same as the previous construction of μ_1 from μ. Define

$$\mathcal{L}_{n,1} = \{ L \in \mathcal{L} : x_n \in L \ , \ x_n + \varepsilon_1 \notin L \}$$

$$a_{n,1} = \inf_{\mathcal{L}_{n,1}} (\mu_{n-1}(L) - \nu(L))$$

$$\mu_{n,1} = \mu_{n-1} + a_{n,1}(\delta_{x_n+\varepsilon_1} - \delta_{x_n})$$

As before $\mu_{n-1} \overset{st}{\leq} \mu_{n,1} \overset{st}{\leq} \nu$ and by Lemma 1 there exists $L_{n,1} \in \mathcal{L}_{n,1}$ such that $\mu_{n,1}(L_{n,1}) = \nu(L_{n,1})$. Continuing inductively, let

$$\mathcal{L}_{n,i} = \{ L \in \mathcal{L} : L \subset L_{n,i-1}, \ x_n \in L, \ x_n + \varepsilon_i \notin L \}$$

$$a_{n,i} = \inf_{\mathcal{L}_{n,i}} (\mu_{n,i-1}(L) - \nu(L))$$

$$\mu_{n,1} = \mu_{n,i-1} + a_{n,i}(\delta_{x_n+\varepsilon_i} - \delta_{x_n})$$

As before $\mu_{n,i-1} \overset{st}{\leq} \mu_{n,i} \overset{st}{\leq} \nu$ and there exists $L_{n,i} \in \mathcal{L}_{n,i}$ such that $\mu_{n,i}(L_{n,i}) = \nu(L_{n,i})$. Define $\mu_n = \mu_{n,k}$, then $\mu_{n-1} \overset{st}{\leq} \mu_n \overset{st}{\leq} \nu$ satisfying (2.11). For $i = k$, we get a set $L_{n,k}$ which satisfies: $x_n \in L_{n,k}$ but $x_n + \varepsilon_i \notin L_{n,k}$ $i = 1, 2, \ldots, k$; this implies that $L_{n,k} \subset \{x_1, x_2, \ldots, x_n\}$, which in turn implies that $L_{n,k} - \{x_n\} \subset \{x_1, x_2, \ldots, x_{n-1}\}$. By construction and inductive hypothesis $\mu_n(\{x_i\}) = \mu_{n-1}(\{x_i\}) = \nu(\{x_i\})$, for $i \leq n-1$, thus $\mu_n(L_{n,k} - \{x_n\}) = \nu(L_{n,k} - \{x_n\})$. However, by definition $L_{n,k}$ is a set which satisfies $\mu_n(L_{n,k}) = \nu(L_{n,k})$. Thus $\mu_n(\{x_n\}) = \nu(\{x_n\})$ verifying (2.12).

Similar to the case of μ and μ_1, the construction of μ_n from μ_{n-1} can be described in terms of a function g_n: $N^k \times [0,1) \to N^k \times [0,1)$ defined as

$$g_n(x_n,z) = \begin{cases} \left(x_n + \varepsilon_i , \dfrac{z - 1 + \sum\limits_{j=1}^{i} a_{n,j}}{a_{n,j}} \right) , & 1-\sum\limits_1^i a_{n,j} \leq z < 1- \sum\limits_1^{i-1} a_{n,j} \\[2ex] \left(x_n , \dfrac{z}{1 - \sum\limits_{j=0}^{k} a_{n,j}} \right) , & 0 \leq z < 1 - \sum\limits_1^k a_{n,j} \end{cases}$$

$$g_n(x,z) = (x,z) \quad , \quad x \neq x_n$$

Then $(\mu_{n-1} \times \lambda) g_n^{-1} = (\mu_n \times \lambda)$ or equivalently $\mu_n = (\mu_{n-1} \times \lambda)$ $(\pi_{N^k} \circ g_n)^{-1}$. Combining this with previous steps gives

$$\mu_n = (\mu\times\lambda)(\pi_{N^k} \circ g_n \circ g_{n-1} \circ \cdots \circ g_1)^{-1}$$

Define f_n: $N^k \times [0,1) \to N^k$ as $f_n = \pi_{N^k} \circ g_n \circ g_{n-1} \circ \cdots \circ g_1$. Thus

$$\mu_n = (\mu\times\lambda)\, f_n^{-1} \tag{2.13}$$

By definition of the g_n's it follows that

$$x << f_1(x,z) << \cdots << f_n(x,z) << \cdots \tag{2.14}$$

for all $(x, z) \in N^k \times [0,1)$, in particular $\{\pi_i f_n(x,z), n = 1, 2, \ldots\}$ is a monotonically increasing sequence as $n \to \infty$ for each $i, i = 1, 2, \ldots, k$. Thus by monotone convergence $\lim_{n\to\infty} f_n(x,z)$ exists for all $(x,z) \in$ $N^k \times [0,1)$.

$$f(x,z) = \lim_{n\to\infty} f_n(x,z) \tag{2.15}$$

If L is a lower set of N^k, it follows from (2.14) and (2.15) that

$$f_1^{-1}(L) \supset f_2^{-1}(L) \supset f_3^{-1}(L) \supset \cdots$$

$$f^{-1}(L) = \bigcap_{n=1}^{\infty} f_n^{-1}(L)$$

and

$$\lim_{n\to\infty} \mu_n(L) = \lim_{n\to\infty} (\mu\times\lambda) f_n^{-1}(L)$$

$$= (\mu\times\lambda)(\cap f_n^{-1}(L))$$

$$= (\mu\times\lambda) f^{-1}(L) \quad .$$

The lower sets are a convergence determining class ([2], p.20) and thus $\mu_n \Rightarrow (\mu\times\nu) f^{-1}$ in the product topology on N^k. However by construction $\mu_n(\{x_m\}) = \nu(\{x_m\})$, $m \le n$ thus $\mu_n \Rightarrow \nu$. Consequently

$$\nu = (\mu\times\lambda) f^{-1} \tag{2.16}$$

Also from (2.14) and (2.15) it follows that, for all $(x,z) \in N^k\times[0,1)$,

$$\pi_{N^k}(x,z) = x << f(x,z). \tag{2.17}$$

Let $\Omega = N^k\times[1,0)$, F = Borel sets, $Q = \mu\times\lambda$, $X' = \pi_{N^k}$, and $Y' = f$, then $Q(X' << Y') = 1$ by (2.17) and X' has measure $(\mu\times\lambda)\pi_{N^k}^{-1} = \mu$ and Y' has measure $(\mu\times\lambda) f^{-1} = \nu$ by (2.16). Thus we have constructed the desired almost surely comparable versions of X and Y.

3. Comparable Random Functions in D(T).

Let T be a bounded or unbounded interval, then $D(T)$ consists of the real-valued functions with domain T which are right continuous and possess left limits. Let the measurable sets of $D(T)$ be those generated by the finite-dimensional cylinder sets ([11], p.62). The partial order on $D(T)$ is the usual one: $f << g$ iff $f(t) \le g(t)$ for all $t \in T$. This section is devoted to proving the following theorem.

Theorem 2. Let X and Y be random functions of $D(T)$ such that $X \overset{st}{\le} Y$ then there exists a probability space (Ω,F,Q) and measurable

functions X', Y': $(\Omega, \mathcal{F}, Q) \to D(T)$ such that $X' \overset{a.s.}{\leq} Y'$ and $X' \overset{D}{=} X$ and $Y' \overset{D}{=} Y$.

The theorem will be proven by approximating X and Y by random vectors in a space homeomorphic to N^k and then using Theorem 1. For this purpose we first prove an approximation lemma.

Lemma 2. Let $x \in D(-\infty, \infty)$, $t_{i,n} = in^{-1}$, $i = 0, \pm 1, \pm 2, \ldots, \pm n^2$.

Define

$$x_n(t) = \begin{cases} -n \vee n \left[\dfrac{x(-n)}{n}\right] & , \ t < -n \\ -n \vee n \left[\dfrac{x(t_{i,n})}{n}\right] & , \ t_{i,n} \leq t < t_{i+1,n} \\ -n \vee n \left[\dfrac{x(n)}{n}\right] & , \ n \leq t \end{cases} \tag{3.1}$$

where $[t]$ signifies greatest integer in t. Then $x_n \overset{J_1}{\to} x$ as $n \to \infty$, where J_1 is the Skorokhod topology on $D(-\infty, \infty)$ as defined by Stone (1963). (We shall denote the function which maps x into x_n in (3.1) as f_n.)

Proof. It follows from Stone's definition of the J_1 topology on $D(-\infty, \infty)$ that $x_n \overset{J_1}{\to} x$ if and only if the restrictions of x_n to a bounded time domain $[-s, s]$ converge in the Skorokhod J_1 topology on $D[-s, s]$ to the restriction of x, for all $s \geq 0$; Whitt (1971, 1972) and Lindvall(1973) expand on this. Thus it suffices to verify J_1 convergence of x_n as random functions of $D[-s, s]$ for arbitrary s. This can be done in a straight-forward manner using criterion 2.6.1 of Skorokhod(1956).

Lemma 3. Let $f_n: D(-\infty, \infty) \to D(-\infty, \infty)$ be the function which maps x into x_n as in (3.1). If X and Y are random functions of $D(-\infty, \infty)$ such that $X \overset{st}{\le} Y$, then $X_n = f_n(X)$ and $Y_n = f_n(Y)$ are random functions of $D(-\infty, \infty)$ such that $X_n \overset{st}{\le} Y_n$.

Proof. If L is a lower set of $D(-\infty, \infty)$ then $f_n^{-1}(L)$ is also a lower set.

Proof of Theorem 2. Approximate X and Y using Lemma 2. Then $X_n = f_n(X)$ and $Y_n = f_n(X)$ will be random functions in a subspace of $D(-\infty, \infty)$ which is homeomorphic to N^{2n^2+1}. Let $h_n: (D(-\infty, \infty), J_1) \to (N^{2n^2+1}$, discrete) be defined: $\pi_i h_n(x_n(\cdot)) = n^2 + nx_n(\frac{i-1}{n} - n)$, $i = 1, 2, 3, \ldots, 2n^2 + 1$, where π_i is the projection. If L is a lower set of N^{2n^2+1} then $h_n^{-1}(L)$ is a lower set of $D(-\infty, \infty)$. This fact and Lemma 3 imply that $h_n(X_n) \overset{st}{\le} h_n(Y_n)$. By theorem 1 there exists a probability space (Ω_n, F_n, Q_n) and measurable functions U_n, V_n: (Ω_n, F_n, Q_n) N^{2n^2+1} such that

$$U_n \overset{D}{=} h(X_n)\ ,\quad V_n \overset{D}{=} h(Y_n)\ ,\quad U_n \overset{a.s.}{\le} V_n \tag{3.2}$$

If we denote $X_n' = h^{-1}(U_n)$ and $Y_n' = h^{-1}(V_n)$ and note that $x << y$ implies $h^{-1}(x) << h^{-1}(y)$ then (3.2) implies

$$X_n' \overset{D}{=} X_n,\ Y_n' \overset{D}{=} Y_n,\quad X_n' \overset{a.s.}{\le} Y_n' \tag{3.3}$$

where $(X_n', Y_n') : (\Omega_n, F_n, Q_n) \to D^2(-\infty, \infty)$, $n = 1, 2, \ldots$.

Now consider the sequence of pairs of random functions, $\{(X_n', Y_n'),\ n = 1, 2, \ldots\}$. From Lemma 2 it follows that $x_n = f_n(x) \overset{J_1}{\to} x$ everywhere, thus

$$X_n = f_n(X) \Rightarrow X\ ,\quad Y_n = f_n(Y) \Rightarrow Y \tag{3.4}$$

in the J_1 topology as $n \to \infty$. Since $X'_n \overset{D}{=} X_n$ and $Y'_n \overset{D}{=} Y_n$, it follows that $\{X'_n, n = 1, 2, \ldots\}$ and $\{Y'_n, n = 1, 2, \ldots\}$ are each relatively J_1-compact in $D(-\infty, \infty)$. Thus given $\epsilon > 0$, there exist J_1-compact subsets K_X, K_Y of $D(-\infty, \infty)$ such that $Q_n(X'_n \in K_X) > 1 - \epsilon/2$ and $Q_n(Y'_n \in K_Y) > 1 - \epsilon/2$ for $n = 1, 2, \ldots$. This implies that $Q_n((X'_n, Y'_n) \in (K_X, K_Y)) > 1 - \epsilon$, but (K_X, K_Y) is compact in $J_1 \times J_1$, thus $\{(X'_n, Y'_n), n = 1, 2, \ldots\}$ is relatively compact and, since the space is metrizable, there must exist a convergent subsequence $(X'_{n'}, Y'_{n'}) \Rightarrow (X', Y')$. $X' \overset{D}{=} X$ and $Y' \overset{D}{=} Y$ follows from (3.3) and (3.4). The subset $\{x << y\}$ of $D(-\infty, \infty) \times D(-\infty, \infty)$ is J_1-closed and thus by the Portmantean theorem ([2], p. 11) $1 = \overline{\lim}_{n\to\infty} P(X'_{n'} << Y'_{n'}) \leq P(X' << Y')$, so $X' \overset{a.s.}{\leq} Y'$ proving the theorem.

<u>Corollary 1</u>. The statement of Theorem 2 also holds for random elements of $D(T)$, $C(T)$, R^∞, and R^k, where T is a bounded or unbounded interval.

<u>Proof</u>. The spaces $D(T)$, $C(T)$, R^∞, and R^k are all either closed subspaces of $D(-\infty, \infty)$ or homeomorphic to closed subspaces of $D(-\infty, \infty)$ (with the homeomorphism preserving the partial order), so random elements of any of these spaces can be considered as random elements of $D(-\infty, \infty)$ to which Theorem 2 then applies.

<u>4. An Equivalent Definition of Stochastic Comparison</u>.

Pledger and Proschan (1973) define stochastic comparison of two random functions $X(\cdot)$ and $Y(\cdot)$ of $D(T)$ as follows: A measurable real-value function on $D(T)$ is increasing if $f(x) \leq f(y)$ for $x << y$.

Then $X \overset{st}{\leq} Y$ if $E(f(X)) \leq E(f(Y))$ for all increasing f.

Since the indicator function of the compliment of a lower set is increasing, their definition is formally stronger than the definition of section 1. However by Theorem 2, if $X \overset{st}{\leq} Y$ according to the definition of section 1, then there exist X', Y' such that $X' \overset{a.s.}{\leq} Y'$. But if $f:D(T) \to R$ is increasing, then $f(X') \overset{a.s.}{\leq} f(Y')$ and consequently $E(f(X)) = E(f(X')) \leq E(f(Y')) = E(f(Y))$ and the two definitions are equivalent.

Barlow and Proschan (1976) use a similar definition to Pledger and Proschan's (1973) definition: they require $E(f(X)) \leq E(f(Y))$ for all increasing f which depend on a finite number of time-coordinates; i.e. for such an f, there exist $t_1, t_2, \ldots, t_n$ such that $f(x(\cdot))$ is determined by $(x(t_1), x(t_2), \ldots, x(t_n))$.

Examining the proofs of Theorems 1 and 2 reveals that it suffices to consider only finite-dimensional lower sets in defining stochastic inequality in order to guarantee the existence of almost surely comparable versions. Thus, as in the preceeding paragraph, the definitions are equivalent. In particular when considering function spaces $D(T)$, $C(T)$, R^∞, R^k with the usual separable topologies, it is equivalent to define stochastic inequality in terms of finite-dimensional lower sets.

REFERENCES

1. Barlow, R.E. and Proschan, F. (1976) Theory of maintained systems: distribution of time to first system failure. Math. Operations Research 1, 32-42.

2. Billingsley, P. (1968) Convergence of Probability Measures. Wiley, New York.

3. Daley, D.J. (1968) Stochastically monotone Markov chains. Z. Wahrscheinlichkeitstheorie verw. Geb. 10, 305-317.

4. Dudley, R.M. (1968) Distances of probability measures and random variables. Ann. Math. Statist. 39, 1563-1572.

5. Dugundji, J. (1966) Topology. Allyn and Bacon, Boston.

6. Jacobs, D.R. and Schach, S. (1972) Stochastic order relationships between GI/G/k systems. Ann. Math. Statist. 43, 1623-1633.

7. Kalmykov, G.I. (1962) On the partial ordering of one-dimensional Markov processes. Theor. Probab. Appl. 7, 456-459.

8. Keilson, J. (1974) Stochastic order in renewal theory and time-reversible chains. Department of Statistics Technical Report No. 82, Stanford University.

9. Keilson, J. and Kester, A. (1974) Monotone matrices and monotone Markov processes. Department of Statistics Technical Report No. 79, Stanford University.

10. Lindvall, T. (1973) Weak convergence of probability measures and random functions in the function space $D[0, \infty)$. J. Appl. Prob. 10, 109-121.

11. Loeve, M. (1963) Probability Theory. Van Nostrand, Princeton.

12. O'Brien, G.L. (1972) A note on comparisons of Markov processes. Ann. Math. Statist. 43, 365-368.

13. O'Brien, G.L. (1975a) The comparison method for stochastic processes. Ann. Probability 3, 80-88.

14. O'Brien, G.L. (1975b) Inequalities for queues with dependent inter-arrival and service times. J. Appl. Probability 12, 653-656.

15. Pledger, G. and Proschan, F. (1973) Stochastic comparisons of random processes with applications in reliability. J. Appl. Probab. 10, 572-585.

16. Proschan, F. and Sethuraman, J. (1975) Stochastic comparisons of order statistics from heterogeneous populations with applications in reliability. Florida State University Technical Report No. M310-R.

17. Pyke, R. (1969) Applications of almost surely convergent constructions of weakly convergent processes. Lecture Notes in Math. 89, 187-200.

18. Ross, S.M. (1974) On time to first failure in multicomponent exponential reliability systems. Operations Research Center Report No. 74-8, University of California, Berkeley.

19. Skorokhod, A.V. (1956) Limit theorems for stochastic processes. Theor. Prob. Appl. 1, 261-290.

20. Sonderman, D. and Whitt, W. (1976) Comparing continuous-time Markov chains. Yale University Technical Report.

21. Stone, C. (1963) Weak convergence of stochastic processes defined on semi-infinite time intervals. Amer. Math. Soc. Proc. 14, 694-696.

22. Stoyan, H. (1973) Monotonie-und Stetigkeitseigenschaften mehrliniger Wartesysteme der Bedienungstheorie. Math. Operationsforsch. u. Statist. 4, no. 2, 155-163.

23. Veinott, A.F. Jr. (1965) Optimal policy in a dynamic, single product, nonstationary inventory model with several demand classes. Operations Research 13, 761-778.

24. Whitt, W. (1971) Weak convergence of probability measures on the function space $D[0, \infty)$. Yale University Technical Report.

25. Whitt, W. (1972) Continuity of several functions on the function space D. Yale University Technical Report.

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